MINIMAL COCYCLES WITH THE SCALING PROPERTY AND SUBSTITUTIONS

BY

JEAN-MARIE DUMONT*

Laboratoire de Mathématiques Discrètes, U.P.R. 9016
Case 930, 163 avenue de Luminy, 13288 Marseille Cedex 9, France
e-mail: dumont@lumimath.univ-mrs.fr

AND

TETURO KAMAE

Osaka City University, Department of Mathematics
Sugimoto 3-3-138, Sumiyoshi-ku, Osaka, 558 Japan
e-mail: h1720@cc.osaka-cu.ac.jp

AND

SATOSHI TAKAHASHI

Osaka University, Department of Mathematics
Machikaneyama-cho 1-1, Toyonaka, Osaka, 560 Japan
e-mail: takahasi@math.wani.osaka-u.ac.jp

ABSTRACT

'Fractal' functions are formulated as a minimal cocycle on a topological dynamics which admits nontrivial scaling transformations. In this paper, it is proved that if in addition it admits a continuous family of scaling transformations, then its **capacity** is not in $o(N^2)$. We define minimal cocycles with nontrivial scaling transformations coming from substitutions on a finite alphabet which are proved to have capacity $O(N)$, so that they admit only a discrete family of scaling transformations. We also construct one which has capacity $O(N^2)$ and admit a continuous family of scaling transformations.

* Supported by C.N.R.S.(U.R.A 225); Partially supported by the Japan Society for the Promotion of Science.
Received March 24, 1992 and in revised form February 15, 1995

1. Introduction

Let Ω be the space of continuous functions $\omega: \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$ with the compact open topology. For $t \in \mathbb{R}$, let us define $U_t: \Omega \to \Omega$ by $(U_t\omega)(s) = \omega(s+t) - \omega(t)$. For $\lambda > 0$, let $V_\lambda: \Omega \to \Omega$ be such that $(V_\lambda\omega)(s) = \lambda^\alpha\omega(\lambda^{-1}s)$, where α is a fixed real number with $0 < \alpha < 1$

We study a nontrivial compact subspace X of Ω with the properties that

(i) $\overline{\{U_t\omega: t \in \mathbb{R}\}} = X$ for any $\omega \in X$, and

(ii) for some positive $\lambda \neq 1$, $V_\lambda X \subset X$.

In this case, the set of λ as above together with 1 is called the **base set** of X. Such X as above is called a **minimal cocycle with the α-scaling property**. If in addition, the base set is $\mathbb{R}_+$, then X is said to have the **continuous** scaling property (otherwise, the **discrete** scaling property).

In this paper, we prove that any function ω in a nonzero minimal cocycle with the α-scaling property is uniformly α-Hölder continuous but nowhere α'-Hölder continuous for any $\alpha' > \alpha$. We also prove that a minimal cocycle with the continuous scaling property cannot have **capacity** in $o(N^2)$. We know examples of minimal cocycles with the continuous scaling property having capacity $O(N^2)$ so that this lower bound is exact. On the other hand, the minimal cocycles determined by substitutions on a finite alphabet have capacity $O(N)$, and hence, have the discrete scaling property.

There are two important aspects of 'fractal' functions: almost periodicity and self-similarity. Our notion of minimal cocycles with the scaling property is a formulation of 'fractal' functions from these points of view.

The self-affine functions in the sense of [1] can be embedded in minimal cocycles with the scaling property. In fact, these functions induce the minimal cocycles determined by substitutions with constant lengths, which are discussed in [2] from the point of view of stochastic processes. The substitutions define counting systems which are generalizations of r-adic representations ([3]). The sums of digits to these counting systems are discussed in [4]. Our dynamical systems defined in Section 3 are translations of the 2-way expanded 'numbers' in these counting systems. In the Japanese text book [5], we have already published most of the results here including a general construction of minimal cocycles with the continuous scaling property. Though a part of the text book was intended to be an introduction to this paper, the delay of its preparation caused the time inversion. Another simpler way of general construction of minimal cocycles with the continuous as well as discrete scaling property together with unique ergodicity is given in a forthcoming paper [6].

2. Minimal cocycles with the scaling property

Let X be a nonempty topological space and $(T_t)_{t\in\mathbb{R}}$ be a **continuous flow** on X, that is,

(1) for any $t \in \mathbb{R}$, T_t is a mapping $X \to X$ such that T_0 is the identity and $T_s \circ T_t = T_{s+t}$ for any $s, t \in \mathbb{R}$, and

(2) $(x,t) \mapsto T_t x$ is a continuous mapping $X \times \mathbb{R} \to X$.

We call a function $F: X \times \mathbb{R} \to \mathbb{R}$ a **cocycle** on X with respect to $(T_t)_{t\in\mathbb{R}}$ if

$$F(x, s+t) = F(x,s) + F(T_s x, t)$$

holds for any $x \in X$ and $s, t \in \mathbb{R}$. It is called a **continuous** cocycle if in addition, $F(x,t)$ is a continuous function of (x,t).

Let Ω be the set of continuous functions $\omega: \mathbb{R} \to \mathbb{R}$ such that $\omega(0) = 0$ with the compact open topology. For any $s \in \mathbb{R}$, we define a mapping $U_s: \Omega \to \Omega$ by

$$(U_s\omega)(t) = \omega(s+t) - \omega(s).$$

Then, $(U_s)_{s\in\mathbb{R}}$ is a continuous flow on Ω.

For a continuous cocycle F on X and $x \in X$, the mapping $t \mapsto F(x,t)$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to Ω, which is denoted by F_x. Then the mapping $x \mapsto F_x$ from X to Ω is continuous. We denote by $\Omega(F)$ the closure of $\{F_x: x \in X\}$. Then, it holds that $U_s\Omega(F) = \Omega(F)$ for any $s \in \mathbb{R}$. The set $\Omega(F)$ is also called a **cocycle**.

A continuous cocycle F is called **minimal** if

(3) $\Omega(F)$ is compact, and

$$\overline{\{U_s\omega: s \in \mathbb{R}\}} = \Omega(F)$$

for any $\omega \in \Omega(F)$.

Let α be a real number with $0 < \alpha < 1$ which we fix throughout this section. A continuous cocycle F is said to have the **α-scaling property** if there exists $\beta > 0$ with $\beta \neq 1$ such that

(4) $V_\beta\Omega(F) \subset \Omega(F)$,

where $V_\beta: \Omega \to \Omega$ is defined as

$$(V_\beta\omega)(t) = \beta^\alpha\omega(\beta^{-1}t) \qquad (\forall t \in \mathbb{R})$$

for any $\omega \in \Omega(F)$.

A positive number β with the property (4) is called a **base** of F with respect to α.

THEOREM 1: *Let F be a nonzero, minimal cocycle with the α-scaling property. Then the following results hold:*

(i) *The set of bases of F with respect to α is a closed, multiplicative subgroup of $\mathbb{R}_+$.*

(ii) *There exists a constant C such that*

$$|\omega(s+t) - \omega(s)| \leq C|t|^\alpha$$

for any $s, t \in \mathbb{R}$ and $\omega \in \Omega(F)$. That is, the functions in $\Omega(F)$ are uniformly α-Hölder continuous.

(iii) *For any $\omega \in \Omega(F)$ and $s \in \mathbb{R}$,*

$$\limsup_{t \downarrow 0} \frac{1}{t^\alpha} |\omega(s+t) - \omega(s)| > 0$$

holds. That is, any $\omega \in \Omega(F)$ is nowhere even locally α'-Hölder continuous for any $\alpha' > \alpha$.

Proof: (i) It is clear by the definition that if β and γ are bases of F with respect to α, then so is $\beta\gamma$. It is also clear that if β is a base with respect to α, then so is β^{-1}, since $V_\beta \Omega(F) = \Omega(F)$ by the minimality of F.

Let β_n's be bases of F with $\beta_n \to \beta$. Then for any $\omega \in \Omega(F)$, it holds that $V_\beta \omega = \lim_{n \to \infty} V_{\beta_n} \omega \in \Omega(F)$. Hence, β is a base. Thus the set of bases is closed.

(ii) By (3),

$$C := \sup_{\omega \in \Omega(F), |t| \leq 1} |\omega(t)| < \infty.$$

Let $\beta > 1$ be a base of F with respect to α. Take any $\omega \in \Omega(F)$ and $s, t \in \mathbb{R}$. Let $k \in \mathbb{Z}$ satisfy that

$$\beta^{k-1} < |t| \leq \beta^k.$$

Then, denoting $\eta = V_{\beta^{-k}} U_s \omega \in \Omega(F)$, we have

$$|\omega(s+t) - \omega(s)| = |(U_s\omega)(t)| = |\beta^{k\alpha} \eta(\beta^{-k} t)| \leq \beta^{k\alpha} C < \beta^\alpha C |t|^\alpha,$$

which proves (ii).

(iii) Let $\beta > 1$ be a base of F with respect to α. We prove that

$$\inf_{\omega\in\Omega(F)}\sup_{1\le t\le\beta}|\omega(t)|>0.$$

Suppose that this is not true. Then, since $\Omega(F)$ is compact, there exists $\omega_0 \in \Omega(F)$ such that $\omega_0(t) = 0$ for any $t \in [1,\beta]$. Since β^n is a base of F with respect to α for any $n = 1,2,\ldots$, there exists $\omega_n \in \Omega(F)$ such that

$$\omega_n(t)=\beta^{n\alpha}\omega_0(\beta^{-n}t)$$

for any $t \in \mathbb{R}$. Then, $\omega_n(t) = 0$ for any $t \in [\beta^n, \beta^{n+1}]$. Let $\omega \in \Omega(F)$ be any limit point of the sequence $(U_{\beta^n+n}\omega_n\colon n = 0,1,2,\ldots)$. Then, it is easy to see that $\omega = 0$. By the minimality, this implies that $\Omega(F) = \{0\}$ and that F is the zero function, which contradicts our assumption.

Thus,

$$\delta:=\inf_{\omega\in\Omega(F)}\sup_{1\le t\le\beta}|\omega(t)|>0.$$

Then, for any $\omega \in \Omega(F)$ and $s \in \mathbb{R}$, it holds that

$$\begin{aligned}
\limsup_{t\downarrow 0}\frac{1}{t^\alpha}|\omega(s+t)-\omega(s)| &= \limsup_{t\downarrow 0}\frac{1}{t^\alpha}|(U_s\omega)(t)| \\
&= \limsup_{n\to\infty}\sup_{1\le t\le\beta}\frac{1}{(\beta^{-n}t)^\alpha}|(U_s\omega)(\beta^{-n}t)| \\
&\ge \beta^{-\alpha}\limsup_{n\to\infty}\sup_{1\le t\le\beta}|\beta^{n\alpha}(U_s\omega)(\beta^{-n}t)| \\
&\ge \beta^{-\alpha}\inf_{\eta\in\Omega(F)}\sup_{1\le t\le\beta}|\eta(t)| \\
&= \beta^{-\alpha}\delta>0,
\end{aligned}$$

which proves (iii). ∎

COROLLARY 1: *Let F be a nonzero minimal cocycle with the α-scaling property. Then α is unique.*

COROLLARY 2: *Let F be a minimal cocycle with the α-scaling property. Let G be the set of bases of F. Then either $G = \mathbb{R}_+$ or there exists $\beta > 1$ such that $G = \{\beta^n\colon n \in \mathbb{Z}\}$.*

THEOREM 2: *Let F be a nonzero, minimal cocycle with the α-scaling property with $0 < \alpha < 1$. Assume that for any $\varepsilon > 0$, the* **capacity**

$$\min\{\sharp\Xi\colon \Xi \subset \Omega(F),\ \textit{for any}\ \omega \in \Omega(F),\ \textit{there exists}\ \eta \in \Xi\ \textit{such that}\ \sup_{0\le t<N} |\omega(t) - \eta(t)| < \varepsilon\}$$

is of $o(N^2)$ as $N \to \infty$. Then there exists $\beta > 1$ such that the set of bases of F is $\{\beta^n\colon n \in \mathbb{Z}\}$.

Proof: Suppose that the conclusion is false. Then, by Corollary 2, the set of bases of F is $\mathbb{R}_+$.

Take any $k = 1, 2, \ldots$. By the assumption, there exists $\Xi_N \subset \Omega(F)$ for any $N = 1, 2, \ldots$ such that for any $\omega \in \Omega(F)$, there exists $\eta \in \Xi_N$ such that

$$\sup_{0\le t<N} |\omega(t) - \eta(t)| < \frac{1}{2k}. \tag{5}$$

Moreover,

$$\lim_{N\to\infty} \frac{\sharp\Xi_N}{N^2} = 0.$$

Take a sequence $(\lambda_N)_{N=1,2,\ldots}$ of positive numbers such that

$$\lim_{N\to\infty} \lambda_N = 0, \quad \lim_{N\to\infty} N\lambda_N = \infty, \quad \text{and} \quad \lim_{N\to\infty} \frac{\sharp\Xi_N}{(N\lambda_N)^2} = 0.$$

Take a sufficiently large N such that $\lambda_N < \frac{1}{6k}$. Let $\omega \in \Omega(F)$,

$$A = \{0, 1, \ldots, [N\lambda_N] - 1\},$$

and

$$B = \{N - [N\lambda_N] + 1, N - [N\lambda_N] + 2, \ldots, N\}.$$

Since the set of bases of F is $\mathbb{R}_+$, for any $(a, b) \in A \times B$, there exists $\eta_{(a,b)} \in \Omega(F)$ such that

$$\eta_{(a,b)}(t) = \left(\frac{N}{b-a}\right)^{\alpha} (U_a\omega)\left(\frac{b-a}{N}t\right) \tag{6}$$

for any $t \in \mathbb{R}$. Since N is sufficiently large, we have

$$\sharp\Xi_N < (N\lambda_N - 1)^2 < [N\lambda_N]^2.$$

Then, there exist $(a,b),(a',b') \in A \times B$ with $(a,b) \neq (a',b')$ such that

$$\sup_{0 \le t < N} |\eta_{(a,b)}(t) - \eta_{(a',b')}(t)| < \frac{1}{k}. \tag{7}$$

Since $(a,b) \neq (a',b')$, either $a \neq a'$ or $b \neq b'$ holds. By symmetry, we may assume that $a \neq a'$ and $a < a'$.

Let $t_0 = 0$ and define t_n inductively by the equation

$$a + \frac{b-a}{N} t_{n+1} = a' + \frac{b'-a'}{N} t_n \qquad (n = 0,1,2,\ldots). \tag{8}$$

Then, by (6) and (8), we have

$$\begin{aligned} &(\omega(s_n) - \omega(s_0)) - (\omega(s_{n+1}) - \omega(s_1)) \\ &\quad = \left(\frac{b-a}{N}\right)^{\alpha} \eta_{(a,b)}(t_n) - \left(\frac{b'-a'}{N}\right)^{\alpha} \eta_{(a',b')}(t_n) \quad (n = 0,1,2,\ldots), \end{aligned} \tag{9}$$

where

$$s_n = a + \frac{b-a}{N} t_n.$$

Since N is sufficiently large, (8) implies that

$$\left|(t_{n+1} - t_n) - \frac{N}{b-a}(a'-a)\right| = \left|\left(\frac{b'-a'}{b-a} - 1\right) t_n\right| \le 3\lambda_N t_n. \tag{10}$$

Moreover since we take a sufficiently large N, $\frac{N}{b-a}$ is sufficiently close to 1, and λ_N is small enough, hence we may assume by (10) that $0 = t_0 < t_1 < \cdots < t_k \le 2k(a'-a) < N$ where k has already been given in (5).

Thus, adding the equations (10), we have for any $n = 0,1,\ldots,k$

$$\left|t_n - \frac{N}{b-a} n(a'-a)\right| \le 6k^2(a'-a)\lambda_N \qquad (n = 0,1,\ldots,k). \tag{11}$$

Since $|\frac{b-a}{N}| \le 1$, this implies that

$$|(s_n - a) - n(a'-a)| \le 6k^2(a'-a)\lambda_N \qquad (n = 0,1,\ldots,k). \tag{12}$$

By (7), (9) and (ii) of Theorem 1, there exists a constant C such that

$$\begin{aligned} &|\omega(s_{n+1}) - \omega(s_n) - \omega(s_1) + \omega(s_0)| \\ \le\ & \left(\frac{b-a}{N}\right)^{\alpha} |\eta_{(a,b)}(t_n) - \eta_{(a',b')}(t_n)| + \left|\left(\frac{b-a}{N}\right)^{\alpha} - \left(\frac{b'-a'}{N}\right)^{\alpha}\right| |\eta_{(a',b')}(t_n)| \\ \le\ & \frac{1}{k} + (1 - (1-2\lambda_N)^{\alpha}) C {t_n}^{\alpha} \le \frac{1}{k} + 2\lambda_N C (2k(a'-a))^{\alpha} \end{aligned}$$

for any $n = 0, 1, \ldots, k$. Therefore, by summing up this inequality for $n = 0, 1, \ldots, m-1$, we have

$$|\omega(s_m) - \omega(s_0) - mA| \le 1 + 2\lambda_N(a' - a)^\alpha 2^\alpha k^{1+\alpha} C$$

for $m = 0, 1, \ldots, k$, where $A = \omega(s_1) - \omega(s_0) = \omega(a') - \omega(a)$. Hence, for $n = 0, 1, \ldots, k$, we have

$$|(U_a\omega)(s_n - a) - nA| \le 1 + 2\lambda_N(a' - a)^\alpha 2^\alpha k^{1+\alpha} C. \tag{13}$$

By (12), (13) and (ii) of Theorem 1,

$$|(U_a\omega)(n(a' - a)) - nA| \le 1 + (a' - a)^\alpha D, \tag{14}$$

where

$$D = (2\lambda_N 2^\alpha k^{1+a} + 6^\alpha \lambda_N{}^\alpha k^{2\alpha})C.$$

By the assumption that the set of bases of F is $\mathbb{R}_+$, there exists $\xi_N \in \Omega(F)$ such that

$$\xi_N(t) = (k(a' - a))^{-\alpha}(U_\alpha\omega)(tk(a' - a))$$

for any $t \in \mathbb{R}$. Then by (14), we have for $n = 0, 1, \ldots, k$,

$$\left|\xi_N\left(\frac{n}{k}\right) - \frac{n}{k}B_N\right| \le k^{-\alpha}(a' - a)^{-\alpha} + k^{-\alpha}D, \tag{15}$$

where

$$B_N = k^{1-\alpha}(a' - a)^{-\alpha}A.$$

Since $\xi_N(\frac{n}{k})$ is uniformly bounded as $N \to \infty$ in $n = 0, 1, \ldots, k$, B_N stays bounded as $N \to \infty$. Let $\{N'\}$ be a subsequence of $\{N\}$ such that there exist $\eta_k \in \Omega(F)$ and $E_k \in \mathbb{R}$ for which

$$\lim_{N' \to \infty} B_{N'} = E_k,$$

and

$$\lim_{N' \to \infty} \xi_{N'}(t) = \eta_k(t)$$

holds uniformly on compacta $t \in \mathbb{R}$. Then since $(a' - a)^{-\alpha} \le 1$,

$$\left|\eta_k\left(\frac{n}{k}\right) - \frac{n}{k}E_k\right| \le k^{-\alpha} \tag{16}$$

follows from (15) for $n = 0, 1, \ldots, k$. Since $\eta_k(\frac{n}{k})$ is uniformly bounded as $k \to \infty$ when $n \leq k$, E_k stays bounded as $k \to \infty$. Then, there exists a subsequence $\{k'\}$ of $\{k\}$ such that there exists $\eta \in \Omega(f)$ and $E \in \mathbb{R}$ for which

$$\lim_{k' \to \infty} E_{k'} = E,$$

and

$$\lim_{k' \to \infty} \eta_{k'}(t) = \eta(t)$$

holds uniformly on compacta $t \in \mathbb{R}$. Then by (16), $\eta(t) = tE$ holds for any $t \in [0, 1]$. This contradicts (iii) of Theorem 1, which completes the proof. ∎

3. Substitutions and cocycles

In this chapter, we construct minimal cocycles with the scaling property determined by substitutions. Before stating the general construction, we give a simpler example.

Let φ be the following substitution on 2 symbols $\{a, b\}$:

$$\varphi(a) = aaabbbaaa \qquad \text{and} \qquad \varphi(b) = bbbaaabbb,$$

where we denote $\varphi(a)_0 = \varphi(a)_1 = \varphi(a)_2 = a$, $\varphi(a)_3 = \varphi(a)_4 = \varphi(a)_5 = b$, etc. Let us consider a formal two-sided expansion in base 9:

$$j = \sum_{-\infty < i < \infty} j_i 9^{-i} \qquad (j_i \in \{0, 1, \ldots, 8\})$$

such that $\liminf_{i \to -\infty} j_i < 8$ and $\limsup_{i \to -\infty} j_i > 0$. For this j, we associate a sequence $\sigma = (\sigma_i)_{i \in \mathbb{Z}}$ on $\{a, b\}$ such that $\varphi(\sigma_i)_{j_i} = \sigma_{i+1}$ for any $i \in \mathbb{Z}$. Let X be the set of such pairs (σ, j). For (σ, j) and (η, h) in X with the property that there exists k such that $\sigma_i = \eta_i$ and $j_i = h_i$ for any $i \leq k$, we can calculate the difference $t := h - j \in \mathbb{R}$ just as the usual 9-adic calculation. In this case, we denote $(\eta, h) = T_t(\sigma, j)$ so that $(T_t)_{t \in \mathbf{R}}$ defines a continuous flow on X, where we identify these (σ, j) and (η, h) if $t = 0$. For $(\sigma, j) \in X$, we define a formal two-sided expansion

$$f(\sigma, j) = \sum_{-\infty < i < \infty} w(\sigma_i) 3^{-i},$$

where $w(a) = 1$ and $w(b) = -1$. Define a cocycle F on X with respect to $(T_t)_{t \in \mathbf{R}}$ by $F((\sigma, j), t) = f(T_t(\sigma, j)) - f(\sigma, j)$, the difference being able to be calculated since the higher digits coincide. This example gives an example of a minimal cocycle with the $\frac{1}{2}$-scaling property.

The general construction is a little more complicated. Let Σ be a finite set with $\sharp\Sigma \geq 2$. Let φ be a substitution on Σ, that is, φ is a mapping from Σ into $\Sigma^+ = \bigcup_{i=1}^{\infty} \Sigma^i$. For $\xi \in \Sigma^+$, we denote by $L(\xi)$ the **length** of ξ, that is $L(\xi) = k$ if and only if $\xi \in \Sigma^k$. An element ξ in Σ^k is denoted as $\xi = \xi_0\xi_1 \cdots \xi_{k-1}$ with $\xi_i \in \Sigma \quad (i = 1, \ldots, k-1)$, so that for $\sigma \in \Sigma$,

$$\varphi(\sigma) = \varphi(\sigma)_0\varphi(\sigma)_1 \cdots \varphi(\sigma)_{L(\varphi(\sigma))-1}.$$

We extend the mapping φ so that the mapping $\varphi\colon \Sigma^+ \to \Sigma^+$ satisfies

$$\varphi(\sigma_0\sigma_1 \cdots \sigma_{k-1}) = \varphi(\sigma_0)\varphi(\sigma_1) \cdots \varphi(\sigma_{k-1})$$

for any $k = 1, 2, \ldots$ and $\sigma_i \in \Sigma \quad (i = 0, 1, \ldots, k-1)$, where the right-hand side implies the concatenation of $\varphi(\sigma_i)$'s. We assume that φ is **mixing**, that is, there exists n such that for any $\sigma, \sigma' \in \Sigma$, σ' appears in $\varphi^n(\sigma)$. Let $\mathbf{M} = (m_{\sigma\sigma'})_{\sigma,\sigma'\in\Sigma}$ be the matrix associated to φ; that is,

$$m_{\sigma\sigma'} = \sharp\{i\colon \varphi(\sigma)_i = \sigma'\}. \tag{17}$$

Since φ is mixing, we have the following results known as the Perron–Frobenius Theorem.

I. There exists a simple eigenvalue λ of $\mathbf{M}$ such that $|\lambda'| < \lambda$ holds for any other eigenvalue λ' of $\mathbf{M}$.

II. There exists a unique row vector $u = (u(\sigma))_{\sigma\in\Sigma}$ and a unique column vector $v = (v(\sigma))_{\sigma\in\Sigma}$ such that

$$u(\sigma) > 0, \quad v(\sigma) > 0 \quad (\forall\sigma \in \Sigma), \quad \sum_{\sigma\in\Sigma} u(\sigma) = 1, \tag{18}$$

$$\sum_{\sigma\in\Sigma} u(\sigma)v(\sigma) = 1, \quad uM = \lambda u \quad \text{and} \quad Mv = \lambda v.$$

Let

$$\underline{X} = \{(\sigma_i, j_i)_{i\in\mathbb{Z}}\colon \sigma_i \in \Sigma, j_i \in \{0, 1, \ldots, L(\varphi(\sigma_i)) - 1\}, \varphi(\sigma_i)_{j_i} = \sigma_{i+1} (\forall i \in \mathbb{Z})\}.$$

We consider $\underline{X}$ the induced topological space as a subset of the product topological space $(\Sigma \times \{0, 1, \ldots, r-1\})^{\mathbb{Z}}$, where $r = \max_{\sigma\in\Sigma} L(\varphi(\sigma))$. Then, $\underline{X}$ is a compact topological space. For $x = (\sigma_i, j_i)_{i\in\mathbb{Z}} \in \underline{X}$ and $n, m \in \mathbb{Z}$ with $n < m$, define a nonnegative integer $\theta(x, n, m)$ inductively as follows:

$$\theta(x,n,n+1) = j_n, \tag{19}$$
$$\theta(x,n,m+1) = \sum_{i<\theta(x,n,m)} L(\varphi(\varphi^{m-n}(\sigma_n)_i)) + j_m.$$

For a subset S of $\mathbb{Z}$, define an equivalence relation Θ_S on $\underline{X}$ by

$$((\sigma_i, j_i)_{i\in\mathbb{Z}}, (\eta_i, h_i)_{i\in\mathbb{Z}}) \in \Theta_S \quad \text{iff } \sigma_i = \eta_i \text{ and } j_i = h_i \text{ for any } i \in S.$$

Let $\Theta = \bigcup_{k\in\mathbb{Z}} \Theta_{(-\infty,k]}$. For $x = (\sigma_i, j_i)_{i\in\mathbb{Z}}$ and $y = (\eta_i, h_i)_{i\in\mathbb{Z}}$ in $\underline{X}$ such that $\sigma_k = \eta_k$ and $\theta(x,k,m) < \theta(y,k,m)$ for some $k, m \in \mathbb{Z}$ with $k < m$, we define

$$\rho_k(x,y) = \lim_{n\to\infty} \lambda^{-n} \sum_{\theta(x,k,n)<i<\theta(y,k,n)} v(\varphi^{n-k}(\sigma_k)_i).$$

It is clear from (18) that the above limit exists. We define $\rho_k(x,x) = 0$ and $\rho_k(y,x) = -\rho_k(x,y)$ for x, y as above. For $(x,y) \in \Theta$, we define $\rho(x,y) = \rho_k(x,y)$ for some $k \in \mathbb{Z}$ with $(x,y) \in \Theta_{(-\infty,k]}$. This definition is independent of the choice of k. It is clear that for any x, y and z in the same equivalence class of Θ, it holds that

$$\rho(x,z) = \rho(x,y) + \rho(y,z). \tag{20}$$

Let κ denote the equivalence relation on $\underline{X}$ such that $(x,y) \in \kappa$ iff $(x,y) \in \Theta$ and $\rho(x,y) = 0$. The equivalence classes of κ consist of either one element or two elements $\{x, x^-\}$; the latter case occurs iff $x = (\sigma_i, j_i)_{i\in\mathbb{Z}} \in \underline{X}$ satisfies that $j_i = 0$ for any sufficiently large $i \in \mathbb{Z}$ and that $j_i > 0$ for some $i \in \mathbb{Z}$. In this case, we define $x^- = (\eta_i, h_i)_{i\in\mathbb{Z}} \in X$ by

$$\eta_i = \sigma_i \text{ and } h_i = j_i \text{ for any } i < k, \tag{21}$$
$$h_k = j_k - 1, \text{ and}$$
$$h_i = L(\varphi(\sigma_i)) - 1 \text{ for any } i > k,$$

where k is the maximum i such that $j_i > 0$. Such an equivalence class $\{x, x^-\}$ is called **rational**. The above k is called the **degree of rationality** of the equivalence class $\{x, x^-\} \in \underline{X}/\kappa$. For $\{x\} \in \underline{X}/\kappa$ which is not rational, the degree of rationality of $\{x\}$ is defined to be ∞, and we identify $\{x\}$ with x. For

$\{x, x^-\} \in \underline{X}/\kappa$ which is rational, we sometimes identify $\{x, x^-\}$ with x. For $x \in \underline{X}/\kappa$ which is identified with $(\sigma_i, j_i)_{i\in\mathbb{Z}} \in \underline{X}$ in this sense, we denote

$$x_{1,i} = \sigma_i \quad \text{and} \quad x_{2,i} = j_i. \tag{22}$$

We may consider ρ as a function on $\underline{X}/\kappa \times \underline{X}/\kappa$. We may also consider Θ_S as a relation on $\underline{X}/\kappa$; for $x, y \in \underline{X}/\kappa$, $(x, y) \in \Theta_S$ iff there exist $\underline{x} \in x$ and $\underline{y} \in y$ such that $(\underline{x}, \underline{y}) \in \Theta_S$. We also consider $\Theta = \bigcup_{k\in\mathbb{Z}} \Theta_{(-\infty,k]}$ as a relation on $\underline{X}/\kappa$. Note that Θ_S is not in general an equivalence relation on $\underline{X}/\kappa$, since the transitivity may fail to hold. However, Θ is an equivalence relation on $\underline{X}/\kappa$. Denote

$$X = \{x \in \underline{X}/\kappa \colon \{\rho(x, y) \colon (x, y) \in \Theta\} = \mathbb{R}\}.$$

We consider the topological space X as a subset of the quotient topological space $\underline{X}/\kappa$. For x and y in X, denote

$$\delta(x, y) = \max\{k \colon (x, y) \in \Theta_{(-k,k)}\} \quad \text{and} \quad d(x, y) = \lambda^{-\max_{z\in X} \delta(x,z)\wedge\delta(z,y)}. \tag{23}$$

Then, it is not difficult to see that d is a metric on X which is consistent with the topology.

It is clear that for any $x \in X$ and $t \in \mathbb{R}$, there exists a unique $y \in X$ such that $\rho(x, y) = t$. This y is denoted as $T_t x$, so that T_t is a transformation on X. By (20), it holds that

$$T_0 = \text{id} \quad \text{and} \quad T_{t+s} = T_t \circ T_s$$

for any t and s in $\mathbb{R}$.

THEOREM 3: *$(T_t)_{t\in\mathbb{R}}$ is a continuous flow on X.*

Proof: It is sufficient to prove that the mapping $(x, t) \mapsto T_t x$ is continuous. Given $x \in X$ and $t \in \mathbb{R}$, take any real number $\varepsilon > 0$. Let $k \in \mathbb{Z}$ satisfy that $(x, T_{\pm(|t|+1)}x) \in \Theta_{(-\infty,k]}$. Take a sufficiently large positive integer M determined later which is larger than both of $-k$ and the minus of the degree of rationality of x. Take any $y \in X$ with $d(x, y) < \lambda^{-M}$. Then, it holds that $\sigma := x_{1,-M} = y_{1,-M}$ (refer to (22)), and

$$|\theta(x, -M, M) - \theta(y, -M, M)| \le 1,$$

where to define θ, we identify x and y with elements in $\underline{X}$ in the previous sense. We can choose M large enough and $\delta > 0$ small enough so that for any $s \in \mathbb{R}$

with $|t-s|<\delta$, we have $(T_tx)_{1,-M}=(T_sy)_{1,-M}=\sigma$ and

$$|\theta(T_tx,-M,M)-\theta(T_sy,-M,M)|\le C_0+2,$$

where

$$C_0=\frac{\max_{\eta\in\Sigma}v(\eta)}{\min_{\eta\in\Sigma}v(\eta)}.$$

Moreover, if M is sufficiently large, this implies that $d(T_tx,T_sy)<\varepsilon$, which completes the proof. ∎

As for the substitution φ, we assume further that the associated matrix $\mathbf{M}$ has an eigenvalue μ such that $1<\mu<\lambda$. Let $w=(w(\sigma))_{\sigma\in\Sigma}$ be a nonzero real column vector such that $\mathbf{M}w=\mu w$. For x and y in X with $(x,y)\in\Theta$, we define $\tau(x,y)$ exactly like $\rho(x,y)$ with μ and w instead of λ and v. Define $F\colon X\times\mathbb{R}\to\mathbb{R}$ by

$$F(x,t)=\tau(x,T_tx). \tag{24}$$

THEOREM 4: *F is a continuous cocycle on X with respect to $(T_t)_{t\in\mathbb{R}}$.*

Proof: Since (20) holds for τ in place of ρ,

$$F(x,s+t)=F(x,s)+F(T_sx,t)$$

holds for any $x\in X$ and $s,t\in\mathbb{R}$. Therefore, to complete the proof, it is sufficient to prove the continuity of $F\colon X\times\mathbb{R}\to\mathbb{R}$. For this purpose, we prove that there exists a constant C such that

$$|F(x,t)|<C|t|^{\alpha} \tag{25}$$

for any $x\in X$ and $t\in\mathbb{R}$, where $\alpha=\frac{\log\mu}{\log\lambda}$. Let $x\in X$ and $t\in\mathbb{R}$. We may assume without loss of generality that $t>0$. An interval $[a,b)$ in $\mathbb{R}$ is called a **(k,σ,x)-interval** if T_ax has the degree of rationality k, $(T_ax)_{1,k}=\sigma$ (refer to (22)) and $b-a=v(\sigma)\lambda^{-k}$. It is also called **$k$-interval** in short. Let k_0 be the minimum value k such that $[0,t)$ contains a k-interval. Then, it holds that $\lambda^{-k_0}\le {C_1}^{-1}t$, where $C_1:=\min_{\sigma\in\Sigma}v(\sigma)$

Let $I_{k_0 0},I_{k_0 1},\dots,I_{k_0,j-1}$ be the sets of k_0-intervals contained in $[0,t)$. Then, it holds that their union becomes one interval and $i_{k_0}:=j\le 2r-2$. Also, $[0,t)$ after subtracting these intervals consists of one or two intervals. Let $I_{k,0},I_{k,1},\dots,I_{k,j'-1}$ be the set of k-intervals contained in the remainder part with $k=k_0+1$.

Then, it holds that $0 \le i_{k_0+1} := j' \le 2r-2$ and the union of these k_0+1-intervals together with the k_0-intervals becomes one interval. By continuing this process we have the decomposition

$$[0,t) \text{ or } (0,t) = \bigcup_{k \ge k_0} \bigcup_{0 \le i < i_k} I_{k,i},$$

where $i_k \le 2r-2$, $I_{k,i}$ is a k-interval ($\forall k \ge k_0$) and the left-hand side is chosen according to whether x is rational or not, respectively. If $[a,b) := I_{k,i}$ is a (k,σ,x)-interval, then by the definition of τ it holds that

$$F(x, I_{k,i}) := F(x,b) - F(x,a) = w(\sigma)\mu^{-k}.$$

Hence, we have

$$\begin{aligned} |F(x,t)| &= |\sum_{k \ge k_0} \sum_{0 \le i < i_k} F(x, I_{k,i})| \\ &\le \sum_{k \ge k_0} \sum_{0 \le i < i_k} C_2 \mu^{-k} \\ &< 2rC_2(1-\mu^{-1})^{-1}\lambda^{-k_0\alpha} \\ &\le 2rC_2(1-\mu^{-1})^{-1}C_1{}^{-\alpha}t^{\alpha} =: Ct^{\alpha}, \end{aligned}$$

where $C_2 := \max_{\sigma \in S} |w(\sigma)|$. Thus, we have (25).

To complete the proof, it is sufficient to prove that $F(x,t)$ is continuous in $x \in X$ for any fixed t. Since $F(x,t) = \tau(x, T_t x)$, we prove the continuity of $\tau(x, T_t x)$ in x. This is just a repetition of the proof of Theorem 3 for τ instead of ρ. ∎

THEOREM 5: *F is minimal.*

Proof: Let

$$\Delta = \{(\sigma,\eta) \in \Sigma \times \Sigma \colon \sigma = \varphi^n(\xi)_i, \eta = \varphi^n(\xi)_{i+1} \text{ for some } n \ge 1, \xi \in \Sigma \text{ and } i \text{ with } 0 \le i < L(\varphi^n(\xi)) - 1\}.$$

Let $\Omega(F)$ be as in Section 2. For any given $T > 0$, define

$$\Omega_T(F) = \{\omega|_{[-T,T]} \colon \omega \in \Omega(F)\},$$

where $\omega|_{[-T,T]}$ is the restriction of the function $\omega\colon \mathbb{R} \to \mathbb{R}$ to $[-T,T]$. Let $k \in \mathbb{Z}$ satisfy

$$\lambda^{-k} \min_{\sigma \in \Sigma} v(\sigma) \ge 2T.$$

For any $(\sigma,\eta)\in\Delta$, choose $x^{\sigma\eta}\in X$ such that $x^{\sigma\eta}=\{y,y^-\}\in \underline{X}/\kappa$ is rational with the degree of rationality $k-1$ in the notation (21) as follows:

$$y^-{}_{1,k}=\sigma \quad\text{and}\quad y_{1,k}=\eta \quad\text{(refer to (22))}.$$

There exists k such that for any $x\in X$, the interval $[-T,T]$ is covered by the union of 2 consecutive k-intervals, say a (k,σ,x)-interval and a (k,η,x)-interval in this order, where we take these 2 intervals even if it is already covered by one of them. Then, it is clear that $F_x|_{[-T,T]}=f_{\sigma,\eta,s}$ for some s with $-v(\sigma)\lambda^{-k}+T\le s\le v(\eta)\lambda^{-k}-T$, where we put $f_{\sigma,\eta,s}=(U_sF_{x^{\sigma\eta}})|_{[-T,T]}$ for $(\sigma,\eta)\in\Delta$ and $x^{\sigma\eta}$ as above. This implies that the set

$$\{f_{\sigma,\eta,s}\colon (\sigma,\eta)\in\Delta, -v(\sigma)\lambda^{-k}+T\le s\le v(\eta)\lambda^{-k}-T\}$$

is dense in $\Omega_T(F)$. Since Δ is a finite set and the set of possible s as above with respect to $(\sigma,\eta)\in\Delta$ is compact, it is easy to see that the above set is closed. Hence we have

$$\Omega_T(F)=\{f_{\sigma,\eta,s}\colon (\sigma,\eta)\in\Delta, -v(\sigma)\lambda^{-k}+T\le s\le v(\eta)\lambda^{-k}-T\}. \tag{26}$$

By the same reason, it is also clear that the functions in $\Omega_T(F)$ are equicontinuous. This proves that $\Omega(F)$ is compact.

Take any $x\in X$ and $f_{\sigma,\eta,s}$ with $(\sigma,\eta)\in\Delta$ and

$$-v(\sigma)\lambda^{-k}+T\le s\le v(\eta)\lambda^{-k}-T.$$

Since φ is mixing, for any $\xi\in\Sigma$ and $i\in\mathbb{Z}$, there exists $K>0$ not depending on x such that $(T_tx)_{1,i}=\xi$ for some $t\in\mathbb{R}$ with $|t|<K$. Let ξ and $n\ge 1$ satisfy that $\varphi^n(\xi)$ contains the block $\sigma\eta$. Then, since there exists t in a bounded set not depending on x such that $(T_tx)_{1,k-n}=\xi$, $(T_tx)_{1,k}$ takes values σ and η consecutively as t increases within a bounded set not depending on x. This implies that there exists $L>0$ not depending on x such that

$$U_tF_x|_{[-T,T]}=f_{\sigma,\eta,s}$$

holds for some $t\in\mathbb{R}$ with $|t|<L$. Since $\Omega(F)$ is the closure of $\{F_x\colon x\in X\}$, this implies that for any $\omega\in\Omega(F)$ and $\omega'\in\Omega(F)$, there exists $t\in\mathbb{R}$ such that $U_t\omega|_{[-T,T]}=\omega'|_{[-T,T]}$. This proves that F is minimal.

THEOREM 6: *F has the α-scaling property with $\alpha = \frac{\log \mu}{\log \lambda}$.*

Proof: For any $x \in X$, define $Sx \in X$ as follows:

$$(Sx)_{i,j} = x_{i,j+1} \quad (\forall i = 1,2; \quad \forall j \in \mathbb{Z}) \quad \text{(refer to (22))}.$$

Then, it is clear from the definition that

$$F(Sx, t) = \lambda^{\alpha} F(x, \lambda^{-1} t) \quad (\forall t \in \mathbb{R}).$$

Since F is minimal, this implies that F has the α-scaling property. ∎

THEOREM 7: *For any $\varepsilon > 0$, it holds for the capacity that*

$$\min\{\sharp\Xi\colon \Xi \subset \Omega(F), \text{ for any } \omega \in \Omega(F), \text{ there exists } \eta \in \Xi \text{ such that } \sup_{0 \le t < N} |\omega(t) - \eta(t)| < \varepsilon\} = O(N)$$

as $N \to \infty$. Hence by Theorem 2, F has the discrete scaling property.

Proof: Since $\Omega(F)$ is a uniformly equicontinuous family, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\omega \in \Omega(F)$ and $t, s \in \mathbb{R}$ with $|t - s| < \delta$, $|\omega(t) - \omega(s)| < \varepsilon$ holds. Take any $N > 0$. Let $T = N$ and apply (26). Let

$$\Xi := \{U_{i\delta} F_{x^{\sigma\eta}}\colon (\sigma, \eta) \in \Delta, \ i = -[N/\delta], -[N/\delta] + l, \ldots, [N/\delta]\}.$$

Then, by (26) and the choice of δ, it holds that for any $\omega \in \Omega(F)$, there exists $\eta \in \Xi$ such that

$$\sup_{0 \le t < N} |\omega(t) - \eta(t)| < \varepsilon.$$

Since $\sharp\Xi = O(N)$ as $N \to \infty$, we complete the proof. ∎

4. Example

Here, we give an example of minimal cocycles with the continuous scaling property for which $O(N^2)$ holds for the capacity. We'll not go into the details since the full proof together with a general construction is given in [5] or [6].

For each α with $0 < \alpha < 1$, there corresponds uniquely λ with $\frac{1}{4} < \lambda < \frac{1}{2}$ by the relation $\lambda^{\alpha} - (\frac{1}{2} - \lambda)^{\alpha} = \frac{1}{2}$. We assume further that λ and $\frac{1}{2} - \lambda$ are multiplicatively independent. Actually this condition is satisfied except for countably many α. Especially, $\alpha = \frac{1}{2}$ satisfies it. Define a continuous function f on $[0, 1]$ as follows:

(i) $f(0)=0$, $f(\lambda)=\lambda^\alpha$, $f(\frac{1}{2})=\frac{1}{2}$,
(ii) $f(\frac{1}{2}+t)=1-f(\frac{1}{2}-t)$ for any $t\in[0,\frac{1}{2}]$,
(iii) $f(\lambda t)=\lambda^\alpha f(t)$ for any $t\in[0,1]$, and
(iv) $f(\lambda+(\frac{1}{2}-\lambda)t)-\lambda^\alpha=-(\frac{1}{2}-\lambda)^\alpha f(t)$ for any $t\in[0,1]$.

Let

$$X:=\{\omega\in\Omega\colon \text{ for any } T>0, \text{ there exists } s \text{ and } B>0 \text{ with } [0,1]\supset[s-B^{-1}T,s+B^{-1}T] \text{ such that } \omega(t)=B^\alpha(f(s+B^{-1}t)-f(s)) \text{ for any } t\in[-T,T]\}.$$

Then, the fact that X is a nonzero minimal cocycle with the continuous α-scaling property follows from the general arguments in [4].

We only show briefly the reason for $O(N^2)$. Take any $\omega\in X$ and $N>0$. Then, for $T=2(\frac{1}{2}-\lambda)^{-2}N$, there exists s and $B>0$ with $[0,1]\supset[s-B^{-1}T,s+B^{-1}T]$ such that $\omega(t)=B^\alpha(f(s+B^{-1}t)-f(s))$ for any $t\in[-T,T]$. Let $A_0:=\{0,1\}$ and $A_{n+1}=\bigcup_{x\in A_n}\{x,x+\lambda y,x+\frac{1}{2}y,x+(1-\lambda)y\}$ $(n=0,1,2,\ldots)$, where for $x\in A_n$, y is the difference from x to the next element in A_n, or $y=0$ if $x=1$. Then, the ratio between the lengths of 2 successive intervals divided by the elements in A_n is either $(\frac{1}{2}-\lambda)/\lambda$, 1 or $\lambda/(\frac{1}{2}-\lambda)$.

Take the minimum n such that for some successive elements a',b',c' in A_n it holds that

$$s-B^{-1}T\le a'\le s<s+B^{-1}N\le c'\le s+B^{-1}T.$$

Then by the definition of f, $f|_{[a',b']}$ and $f|_{[b',c']}$ are $\pm f$ with scaling transformations on variables and functions. Define j and k to be ±1 corresponding to the above $\pm$ in $\pm f$ for $f|_{[a',b']}$ and $f|_{[b',c']}$, respectively. Let $a=B(a'-s)$, $b=B(b'-s)$ and $c=B(c'-s)$. Let $e=(c'-b')/(b'-a')$. Then, $\omega|_{[0,N]}$ can be reproduced by the following information:

$$\text{real numbers } a \text{ and } b,\ j \text{ and } k \text{ in } \{-1,1\},\quad e\in\left\{\frac{\frac{1}{2}-\lambda}{\lambda},1,\frac{\lambda}{\frac{1}{2}-\lambda}\right\},$$

where $b-a=O(N)$ and $[a,c]\supset[0,N]$ with $c=b+(b-a)e$.

The way of the construction is as follows. Define functions $g_1\colon[a,b]\to\mathbb{R}$ and $g_2\colon[b,c]\to\mathbb{R}$ by

$$g_1(t)=j(b-a)^\alpha f\left(\frac{t-a}{b-a}\right),\qquad g_2(t)=g_1(b)+k(c-b)^\alpha f\left(\frac{t-b}{c-b}\right).$$

Let g: $[a,c] \to \mathbb{R}$ be g_1 on $[a,b]$ and g_2 on $[b,c]$. Then, we have $\omega(t) = g(t) - g(0)$ for any $t \in [0,N]$. Since g is uniformly α-Hölder continuous, to get an ε-approximation of $\omega|_{[0,N]}$, it is sufficient to have $C\varepsilon^{1/\alpha}$-approximations of a and b together with j, k and e. Therefore, $O(N^2)$ number of different g's are enough to approximate any of $\omega|_{[0,N]}$ up to ε by one of g's as $N \to \infty$ and ε fixed. Thus, we have $O(N^2)$ for the capacity of X.

ACKNOWLEDGEMENT: The authors thank the referee for many useful suggestions which make the paper more understandable.

References

[1] T. Kamae, *A characterization of self-affine functions*, Japan Journal of Applied Mathematics **3-2** (1986), 271–280.

[2] T. Kamae and M. Keane, *A class of deterministic self-affine processes*, Japan Journal of Applied Mathematics **7-2** (1990), 185–195.

[3] J-M. Dumont and A. Thomas, *Systemes de numeration et functions fractales relatifs aux substitutions*, Theoretical Computer Science **65** (1989), 153–169.

[4] J-M. Dumont and A. Thomas, *Digital sum moments and substitutions*, Acta Arithmetica **44** (1993), 205–225.

[5] T. Kamae and S. Takahashi, *Ergodic Theory and Fractals*, Springer-Verlag, Tokyo, 1993 (in Japanese).

[6] T. Kamae, *Linear expansions, strictly ergodic homogeneous cocycles and fractals*, to appear.